Correlation-tensor criteria for genuine multiqubit entanglement

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We present a development of a geometric approach to entanglement indicators. The method is applied to detect genuine multiqubit entanglement. The criteria are given in the form of nonlinear conditions imposed on correlation tensors. Thus they involve directly observable quantities, and in some cases require only few specific measurements to find multiqubit entanglement. The nonlinearity of each of the criteria allows detection of entanglement in wide classes of states. In contrast to entanglement witnesses, which in the space of Hermitian operators define a hyperplane, these conditions define a geometric figure encapsulating the nonfully entangled states within it.

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I. INTRODUCTION

Since quantum entanglement is both a basic resource in quantum communication and quantum information processing, and a fundamental phenomenon in considerations related to foundations of quantum physics, its qualitative and quantitative characterization becomes of great importance for practical as well as purely theoretical reasons [1,2].

In contrast to the bipartite case, where the structure of entanglement is very simple (the state is either entangled or separable), for many subsystems the characterization of entanglement becomes more complex due to many possible ways of partitioning the whole system into subsystems. Several different approaches to detect genuine multipartite entanglement have been proposed: based on Bell inequalities [3–7], using entanglement witnesses [8,9], based on relations between elements of density matrices [10,11], utilizing Fisher information [12,13], and finally, using correlation tensors [14].

In this contribution we further develop a geometric approach to entanglement detection proposed in Ref. [15]. We show how this approach leads to necessary and sufficient conditions for various forms of multipartite entanglement and derive explicit criteria for different types of nonseparability. The resulting conditions are of the form of nonlinear combinations of correlation functions, and sometimes the criteria are very simple and require only a limited number of specific directly measurable data. The nonlinearity of our criteria makes them often more versatile than entanglement witnesses. We provide examples in which our criterion detects genuine multipartite entanglement of different families of quantum states. In our investigations, an inspiring role was played by the work of Yu *et al.* [16], in which a general nonlinear condition for two-qubit entanglement was found.

II. REPRESENTATION OF STATES IN TERMS OF CORRELATIONS

Any n-qubit state can be expressed as

$$\rho = \frac{1}{2^n} \sum_{\mu_1,\ldots,\mu_n=0,1,2,3} T_{\mu_1,\ldots,\mu_n} \sigma_{\mu_1} \otimes \cdots \otimes \sigma_{\mu_n}, \tag{1}$$

where $\sigma_{\mu_k} \in \{\mathbb{1},\sigma_1,\sigma_2,\sigma_3\}$ are the Pauli matrices of the kth observer. The coefficients $T_{\mu_1,\ldots,\mu_n}$ are real numbers in $[-1,1]$ given by correlation function values for measurements of products of Pauli operators

$$T_{\mu_1,\ldots,\mu_n} = \langle \sigma_{\mu_1} \otimes \cdots \otimes \sigma_{\mu_n} \rangle_\rho = \mathrm{Tr}\big(\rho\, \sigma_{\mu_1} \otimes \cdots \otimes \sigma_{\mu_n}\big). \tag{2}$$

A specific role is played by the components which involve only indices 1,2,3 (such indices will be denoted by Latin letters). The quantity

$$\hat{T} \equiv \sum_{i_1,\ldots,i_n=1}^{3} T_{i_1,\ldots,i_n} e^{i_1} \otimes \cdots \otimes e^{i_n}, \tag{3}$$

where $\{e^{i_m}\}_{i_m=1}^3$ is a basis in $\mathbb{R}^3$, transforms like a tensor under local unitary transformations on the qubits. As a consequence, we refer to it as a correlation tensor of the state ρ. The whole object $T \equiv T_{\mu_1,\ldots,\mu_n}$, where the indices take on values $\mu_k = 0,1,2,3$, will be called an extended correlation tensor. Its components with k zeros are $(n-k)$-rank tensors. Extended correlation tensors belong to a real vector space with a natural scalar product

$$(X,Y) = \sum_{\vec{\mu}} X_{\vec{\mu}} Y_{\vec{\mu}}, \tag{4}$$

where $\vec{\mu} = (\mu_1,\ldots,\mu_n)$ and $\mu_k = 0,1,2,3$. We can generalize the notion of a scalar product between correlation tensors introducing a positive semidefinite metric G. A generalized scalar product has then the following form:

$$(X,Y)_G = \sum_{\vec{\mu},\vec{\nu}} X_{\vec{\mu}} G_{\vec{\mu}\vec{\nu}} Y_{\vec{\nu}}. \tag{5}$$

This scalar product induces a G norm:

$$||T||_G^2 = (T,T)_G. \tag{6}$$

Such a generalized scalar product and G norm were used in Ref. [15].

III. MULTIPARTITE ENTANGLEMENT

Let us begin with a classification of entanglement of multipartite states. A pure n-partite state $|\psi\rangle$ is called a k product, if it can be represented as a tensor product of k pure i_m-partite states:

$$|\psi_{k\ \mathrm{prod}}\rangle = |\psi_{i_1}\rangle \otimes \cdots \otimes |\psi_{i_k}\rangle, \tag{7}$$

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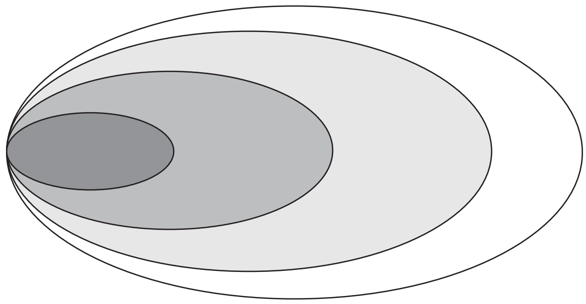

FIG. 1. Sets of k-separable states: all sets are convex; the darker the color of a set, the more separable are its states, i.e., a set of k-separable states contains as its subsets all more-than-k-separable states.

where of course, $\sum_{m=1}^{k} i_m = n$. There can be different types of k-product states corresponding to different ways of partitioning n into a sum of k integers. We will refer to a definite type of k-product state as the $(i_1 + \cdots + i_k)$-partition product state. Note that different k-product states of the same type may involve different physical subsystems in their partitions. For example, $(2+1)$-partition product states of three particles A, B, and C are $|\psi_A\rangle\,|\psi_{BC}\rangle$, $|\psi_B\rangle\,|\psi_{AC}\rangle$, and $|\psi_C\rangle\,|\psi_{AB}\rangle$. We will refer to these types of states as $(A+BC)$-, $(B+AC)$-, and $(AB+C)$-product states, respectively.

The n-partite state ρ is called k separable, if it can be expressed as a probabilistic (convex) mixture of pure k-product states:

$$\rho_{k\text{sep}} = \sum_i p_i \big|\psi^i_{k\text{ prod}}\big\rangle\big\langle\psi^i_{k\text{ prod}}\big|. \tag{8}$$

In this terminology, fully separable states are n separable. If a state is not k separable, then it must involve entanglement between at least $n-k+2$ parties. Accordingly, a state is called genuinely multipartite entangled, if it is not biseparable, i.e., not 2 separable.

Clearly, a set of k-separable states is convex and is a subset of $(k-1)$-separable states, as illustrated in Fig. 1. Note that, infinitesimally close to pure fully separable states there are states with entanglement between an arbitrary number of subsystems. For an illustrative example, see Sec. V B.

IV. NECESSARY AND SUFFICIENT CONDITIONS FOR MULTIPARTITE ENTANGLEMENT

To indicate a case of a non-k separability we shall use the following simple geometrical observation, which is a corollary of the results of [15], namely, that

$$\Big(\exists_G \max_{T^{k\text{-sep}}} (T^{k\text{-sep}}, T)_G < ||T||^2_G\Big) \Longrightarrow T \text{ is not } k \text{ separable}, \tag{9}$$

where $T^{k\text{-sep}}$ is a correlation tensor of some k-separable state, and G is a metric in the sense of Eq. (5). It forms a sufficient condition for multipartite entanglement between at least $n-k+2$ parties.

The advantage of using scalar products to entanglement detection is that optimization over separable states in Eq. (9) can be replaced with optimization over pure k-product states only:

$$\max_{T^{k\text{-sep}}} (T^{k\text{-sep}}, T)_G = \max_{\{\{p_i\}, T^{k\text{-prod}}\}} \left(\sum_i p_i T^{k\text{-prod}}_{(i)}, T\right)_G \leqslant \max_{T^{k\text{-prod}}} (T^{k\text{-prod}}, T)_G. \tag{10}$$

This follows directly from linearity of a scalar product and convexity of k-separable states. Since k-separable states may involve different types of k-product states, one should optimize over all possible partitions compatible with k separability. This reveals another feature of our condition of potential practical value: for different partitions π one can use different metrics to reveal that the state is not k product. All of the above implies the following modified condition:

$$\Big(\forall_\pi \exists_{G_\pi} \max_{T_\pi^{k\text{-prod}}} \big(T_\pi^{k\text{-prod}}, T\big)_{G_\pi} < ||T||^2_{G_\pi}\Big) \Longrightarrow T \text{ is not } k \text{ separable}. \tag{11}$$

Finally, it is easy to show by adapting the reasoning of [15], that in case of any non-k-separable multiqubit state, for any given partition, one can find such a metric $\tilde{G}$ that the left-hand side of the inequality in condition (11) holds, which leads to a necessary and sufficient condition for genuine multipartite entanglement. Indeed, the condition of rejecting full separability, originally formulated in Ref. [15] in terms of density matrices, can be set as follows:

$$\max_{\rho_{\text{prod}}} \text{Tr}(\rho\tilde{G}\rho_{\text{prod}}) < \text{Tr}(\rho\tilde{G}\rho) \Longrightarrow \rho \text{ is not a product state}, \tag{12}$$

where now $\tilde{G}$ is a positive semidefinite superoperator. In Ref. [15] it is shown that, conversely, if a state ρ is not separable, there exists a positive semidefinite superoperator $\tilde{G}$, such that the inequality (12) holds. To state this fact clearly, let us introduce new notation. Let $\{f_\mu\}_{\mu=0}^{3}$ denote the standard basis in the space of 2×2 matrices over complex numbers $\mathbb{M}_2(\mathbb{C})$, which is the space of operators acting on a Hilbert space $\mathbb{C}^2$ of a single qubit. Any density matrix ρ can then be decomposed as $\rho = \sum_{\vec{\mu}} \rho_{\vec{\mu}} f_{\vec{\mu}}$, where $\vec{\mu} = \{\mu_1, \ldots, \mu_n\}$ and $f_{\vec{\mu}} = f_{\mu_1} \otimes \cdots \otimes f_{\mu_n}$ is a basis in a tensor product space $(\mathbb{M}_2)^{\otimes n} = \mathbb{M}_{2^n}$. According to [15], if a state ρ is not separable, then there exists a superoperator $\tilde{G}$, such that

$$\max_{\rho_{\text{prod}}} \text{Tr}(\rho\tilde{G}\rho_{\text{prod}}) < \text{Tr}(\rho\tilde{G}\rho). \tag{13}$$

In the case of rejecting k separability instead of full separability we proceed in full analogy, except for the fact that for optimizing the left-hand side of Eq. (12) for each partition, we treat all elements of this partition as a single system (of course it can be of higher dimension). Having k single systems we can find a metric $\tilde{G}$ for each partition separately, since the correctness of condition (13) does not depend on the dimensions of the single systems under consideration. By choosing a specific basis (setting $f_\mu = \sigma_\mu$), all the above considerations can be translated into correlation tensor representation of a state (1). Indeed, having a density matrix of any state in correlation tensor form (1), we can find its components in a standard basis in the following manner:

$$\rho_{\vec{\mu}} = \sum_{\vec{\nu}} T_{\vec{\nu}} U_{\vec{\mu}\vec{\nu}}. \tag{14}$$

The scalar product between density matrices ρ and ρ' in metric $\tilde{G}$ can now be reformulated as a scalar product between extended correlation tensors:

$$\begin{aligned}\mathrm{Tr}(\rho\tilde{G}\rho') &= (\rho,\rho')_{\tilde{G}} = \sum_{\vec{\mu}\vec{\nu}} \rho_{\vec{\mu}} \tilde{G}_{\vec{\mu}\vec{\nu}} \rho'_{\vec{\nu}} \\ &= \sum_{\vec{\mu}\vec{\nu}} \sum_{\vec{\gamma}\vec{\delta}} T_{\vec{\gamma}} U_{\vec{\mu}\vec{\gamma}} \tilde{G}_{\vec{\mu}\vec{\nu}} T'_{\vec{\delta}} U_{\vec{\nu}\vec{\delta}} \\ &= \sum_{\vec{\gamma}\vec{\delta}} T_{\vec{\gamma}} G_{\vec{\gamma}\vec{\delta}} T'_{\vec{\delta}} = (T,T')_G.\end{aligned} \tag{15}$$

The above transformation implies that the metric operator G, which defines a scalar product of correlation tensors [cf. Eq. (5)] corresponding to a scalar product of density matrices with superoperator $\tilde{G}$, has the following form:

$$G_{\vec{\gamma}\vec{\delta}} = \sum_{\vec{\mu}\vec{\nu}} U_{\vec{\mu}\vec{\gamma}} \tilde{G}_{\vec{\mu}\vec{\nu}} U_{\vec{\nu}\vec{\delta}}. \tag{16}$$

Altogether we have shown the following proposition fully characterizing k separability.

Proposition 1. An n-particle state endowed with extended correlation tensor T is not k separable if and only if for every partition π into k subsystems there exists a metric G_π, such that the following inequality holds:

$$\max_{T_\pi^{k-\mathrm{prod}}} \left(T_\pi^{k-\mathrm{prod}},T\right)_{G_\pi} < (T,T)_{G_\pi}. \tag{17}$$

V. EXAMPLES

We present here a few examples of applications of Proposition 1. It leads to simple sufficient conditions for multipartite entanglement that are sometimes also necessary and detect entanglement of various classes of states (which is impossible using entanglement witnesses).

A. Three-qubit entanglement

To reveal a genuine three-partite entanglement in a three-qubit state we have to exclude the case of biseparability. We derive several sufficient criteria using different metric tensors.

1. Standard metric

Here we shall give a condition which is unbiased in its formulation with respect to any family of entangled states. It uses the diagonal metric, that with $G_{\vec{\mu}\vec{\nu}}$ in the form of a Kronecker delta $\delta_{\vec{\mu}\vec{\nu}}$.

Let $\hat{T}$ be a correlation tensor of a three-qubit state, and let $\hat{T}^{2+1}$ be a correlation tensor of a $(2+1)$-partition product three-qubit state. Assuming a standard (Euclidean) scalar product in space $\mathbb{R}^9$,

$$(\hat{X},\hat{Y}) = \sum_{i,j,k=1}^{3} X_{ijk} Y_{ijk}, \tag{18}$$

we show that $\max_\sigma \max_{\hat{T}_\sigma^{2+1}} (\hat{T}_\sigma^{2+1},\hat{T})$, where σ denotes permutation of biproduct states (here three possible splittings), is upper bounded by

$$\max_\sigma \max_{\sigma(\hat{O}\otimes\hat{O}',\hat{\mathbb{1}})} \sqrt{\sum_{i=1}^{3} (|T_{\sigma(11i)} - T_{\sigma(22i)}| + |T_{\sigma(33i)}|)^2}, \tag{19}$$

where $\sigma(\hat{O}\otimes\hat{O}',\hat{\mathbb{1}})$ means that the second maximization is done over local orthogonal transformations applied to subsystems over which, for a given σ, summation is not performed, e.g., if σ is a trivial permutation, $\sigma(\hat{O}\otimes\hat{O}',\hat{\mathbb{1}}) = \hat{O}\otimes\hat{O}'\otimes\hat{\mathbb{1}}$.

Clearly $\hat{T}^{2+1} = \hat{T}^2 \otimes \hat{T}^1$. Any pure two-qubit state can be expressed in a Schmidt basis in the form $\cos\theta|00\rangle + \sin\theta|11\rangle$, which has the following nonvanishing terms of correlation tensor $\hat{T}^2$:

$$\begin{aligned} T_{11} &= \sin 2\theta, \\ T_{22} &= -\sin 2\theta, \\ T_{33} &= 1, \end{aligned} \tag{20}$$

while $\hat{T}^1$ is a Bloch vector:

$$\hat{T}^1 = \vec{m} = [m_1,m_2,m_3], \quad \text{with } \sqrt{m_1^2+m_2^2+m_3^2} = 1. \tag{21}$$

Hence the scalar product $(\hat{T}_\sigma^{2+1},\hat{T})$, where σ denotes some permutation of indices defining to which subsystems tensors $\hat{T}^2$ and $\hat{T}^1$ correspond, can be expressed as

$$\left(\hat{T}_\sigma^{2+1},\hat{T}\right) = \sum_{i=1}^{3} [(T_{\sigma(11i)} - T_{\sigma(22i)})\sin 2\theta + T_{\sigma(33i)}]m_i. \tag{22}$$

Since $\vec{m}$ is an arbitrary unit vector, the maximization over $\vec{m}$ gives

$$\max_{\vec{m}} \left(\hat{T}_\sigma^{2+1},\hat{T}\right) = \sqrt{\sum_{i=1}^{3} [(T_{\sigma(11i)} - T_{\sigma(22i)})\sin 2\theta + T_{\sigma(33i)}]^2}. \tag{23}$$

Finally we have to maximize over θ, over permutations σ of subsystems and over all possible local rotations $\hat{O}\otimes\hat{O}'$ applied to subsystems, over which we do not sum in Eq. (23). At this stage we do not need to maximize over rotations applied to the subsystem over which we sum in the above equation, since maximization over a unit $\vec{m}$ is equivalent to maximization over all possible rotations of this vector, and the maximization over rotations of the scalar product $(\hat{T}_\sigma^{2+1},\hat{T})$ can be performed over any of the tensors in this product.

Adopting the notation introduced in Eq. (19) and using the fact that for reals r_1 and r_2 we have $(r_1 \sin 2\theta + r_2)^2 \leqslant (|r_1| + |r_2|)^2$, we reach the following condition.

Proposition 2. If the following inequality holds,

$$\max_{\sigma,\sigma(\hat{O}\otimes\hat{O}',\hat{\mathbb{1}})} \sqrt{\sum_{i=1}^{3} (|T_{\sigma(11i)} - T_{\sigma(22i)}| + |T_{\sigma(33i)}|)^2} < ||\hat{T}||^2, \tag{24}$$

then the state described by correlation tensor $\hat{T}$ is genuinely three-partite entangled.

However, as the left-hand side of this proposition may be strictly bigger than Eq. (23), in some cases it is more effective to directly use condition

$$\max_{\sigma} \max_{\hat{T}_{\sigma}^{2+1}} \left(\hat{T}_{\sigma}^{2+1}, \hat{T}\right) \leqslant ||\hat{T}||^2. \tag{25}$$

For example, for a generalized three-partite GHZ state

$$|\mathrm{GHZ}_\alpha\rangle = \cos\alpha|000\rangle + \sin\alpha|111\rangle, \tag{26}$$

mixed with white noise,

$$v|\mathrm{GHZ}_\alpha\rangle\langle\mathrm{GHZ}_\alpha| + (1-v)\tfrac{1}{8}\mathbb{1}, \tag{27}$$

Eq. (23) is maximized for vanishing θ, and reads $v\sqrt{1+3\sin^2 2\alpha}$. The same value is obtained after applying local rotations as we verified numerically. Since the squared length of the correlation tensor equals $v^2(1+3\sin^2 2\alpha)$, we find that if v exceeds the critical value

$$v_{\mathrm{crit}} = \frac{1}{\sqrt{1+3\sin^2 2\alpha}}, \tag{28}$$

the state is genuinely three-partite entangled for any α. Local rotations applied to the left-hand side of Eq. (24) make its value higher than the one which follows from Eq. (23), and accordingly, Proposition 2 does not detect as many states as the direct application of Eq. (23). Finally, note that the critical visibility (28) holds for all possible locally unitarily equivalent three-partite GHZ states. For a symmetric GHZ state (that is, for $\alpha = \pi/4$), $v_{\mathrm{crit}} = \frac{1}{2}$. Note that even for an arbitrarily small but finite α, the state (26) is three-partite entangled, while for $\alpha = 0$ it is fully separable.

In the case of a three-partite W state $|W_3\rangle = \frac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle)$ mixed with white noise,

$$\rho_W(p) = v|W_3\rangle\langle W_3| + (1-v)\tfrac{1}{8}\mathbb{1}, \tag{29}$$

the critical visibility for detection of the genuine three-qubit entanglement with condition (24) has been calculated numerically, and is equal to $v_{\mathrm{crit}} \approx 0.636$.

We stress that this approach is quite versatile; it allows one to detect three-particle entanglement despite the fact that the GHZ and W states are of a different nature.

2. Modified metric and generalized Schmidt decomposition of the correlation tensor

The criterion in Proposition 2 can be made more efficient and simplified by changing a metric and applying a generalized Schmidt decomposition [17] to the correlation tensor.

In order to arrive at a simple criterion, we modify the metric [hence also the scalar product in the sense of Eq. (5)] to get rid of terms of the $T_{\sigma(33i)}$ type. Thus, we set

$$||\hat{T}||^2_{\mathrm{mod}} = \sum_{i,j,k=1}^{3} T_{ijk}^2 - \sum_{l=1}^{3} T_{\sigma(33l)}^2. \tag{30}$$

Using this metric, the inequality (24) can be rewritten as

$$\max_{\sigma} \max_{\sigma(\hat{O}\otimes\hat{O}',\hat{\mathbb{1}})} \sqrt{\sum_{i=1}^{3}(T_{\sigma(11i)} - T_{\sigma(22i)})^2} < ||\hat{T}||^2_{\mathrm{mod}}. \tag{31}$$

Since the condition (9) is valid in any metric G, we have the following modified criterion.

Proposition 3. If the following inequality holds,

$$\max_{\sigma} \max_{\sigma(\hat{O}\otimes\hat{O}',\hat{\mathbb{1}})} \sqrt{\sum_{i=1}^{3}(T_{\sigma(11i)} - T_{\sigma(22i)})^2} < ||\hat{T}||^2_{\mathrm{mod}}, \tag{32}$$

where $||\hat{T}||^2_{\mathrm{mod}} = \sum_{i,j,k=1}^{3} T_{ijk}^2 - \sum_{l=1}^{3} T_{\sigma(33l)}^2$, then the state described by correlation tensor $\hat{T}$ is genuinely three-partite entangled.

We can further simplify this condition by applying handy features of a generalized Schmidt decomposition to the correlation tensor [17]. According to Theorem 1 in Ref. [17], for any tensor

$$\hat{T} = T_{i_1,\ldots,i_n} e^1_{i_1} \otimes \cdots \otimes e^n_{i_n}, \quad \text{with } i_k = 1,\ldots,d, \tag{33}$$

where $\{e^m_{i_m}\}_{i_m=1}^{d}$ is a basis in some d-dimensional vector space, there exists a basis $s^1_{i_1} \otimes \cdots \otimes s^n_{i_n}, i_k = 1,\ldots,d$, which we shall call a generalized Schmidt basis, in which the components $T'_{i_1,\ldots,i_n}$ of tensor $\hat{T}$,

$$\hat{T} = T'_{i_1,\ldots,i_n} s^1_{i_1} \otimes \cdots \otimes s^n_{i_n}, \tag{34}$$

have the following properties:

$$T'_{\sigma(j,i,\ldots,i)} = 0 \text{ for } 1 \leqslant i < j \leqslant d, \tag{35}$$

$$T'_{i_1,\ldots,i_n} \text{ is non-negative if at most one of the } i_k \text{ differs from } d, \tag{36}$$

$$|T'_{j,\ldots,j}| \geqslant |T'_{i_1,\ldots,i_n}| \text{ if } j \leqslant i_r \text{ for all } r = 1,\ldots,n. \tag{37}$$

Assume that the correlation tensor $\hat{T}$ in Eq. (32) is expressed in a generalized Schmidt basis. Then the property (35) implies, that from among the following two groups of terms of the correlation tensor,

$$\{T_{111}, T_{112}, T_{113}\}, \quad \{T_{221}, T_{222}, T_{223}\},$$

only one term in each group is nonzero. Let us assume without losing generality that T_{111} is the maximal generalized Schmidt coefficient and only T_{221} is nonzero. This implies that for all σ,

$$\sqrt{\sum_{i=1}^{3}(T_{\sigma(11i)} - T_{\sigma(22i)})^2} \leqslant |T_{111}| + |T_{221}| \leqslant 2|T_{111}|.$$

This property gives the following criterion.

Proposition 4. If the following inequality holds,

$$||\hat{T}||^2_{\mathrm{mod}} > 2T_{\max}, \tag{38}$$

where $||\hat{T}||^2_{\mathrm{mod}} = \sum_{i,j,k=1}^{3} T_{ijk}^2 - \sum_{l=1}^{3} T_{\sigma(33l)}^2$, and $T_{\max}$ is the maximal possible value of a correlation tensor element for the given state, then the state described by correlation tensor $\hat{T}$ is genuinely three-partite entangled.

The above condition can be simplified further to a weaker one:

$$||\hat{T}||^2_{\mathrm{mod}} > 2,$$

which detects a smaller class of entangled states, but is experimentally very handy. One can measure components of

$\hat{T}$ which enter $||\hat{T}||^2_{\text{mod}}$, and once the sum (30) is above 2, a genuine three-particle entanglement is confirmed.

All of the above analysis can be performed in full analogy in the case of four-partite states. Due to the complexity of the formulas, conditions for the four-qubit entanglement are presented in the Appendix.

B. GHZ metric

In this section we shall study an approach that favors a certain family of states, namely, the GHZ states.

Consider the so-called Greenberger-Horne-Zeilinger state of n qubits:

$$|\text{GHZ}\rangle = \frac{1}{\sqrt{2}}\left(|0\cdots 0\rangle + |1\cdots 1\rangle\right). \tag{39}$$

It has the following nonvanishing elements of extended correlation tensor [18]:

$$\begin{aligned} T^{\text{GHZ}}_{\underbrace{y\cdots y}_{2k}x\cdots x} &= (-1)^k, \qquad k = 0,1,\ldots,\lfloor\tfrac{n-1}{2}\rfloor, \\ T^{\text{GHZ}}_{\underbrace{z\cdots z}_{2k}0\cdots 0} &= 1, \end{aligned} \tag{40}$$

and with indices permuted. It turns out that Proposition 1 with a diagonal metric G,

$$(X,Y)_G = \sum_{\vec{\mu}} X_{\vec{\mu}} G_{\vec{\mu}} Y_{\vec{\mu}}, \tag{41}$$

for the specific case of $G_{\vec{\mu}} = |T^{\text{GHZ}}_{\vec{\mu}}|$ and additionally setting $G_{0,\ldots,0} = 0$, leads to optimal detection of a genuine multipartite entanglement of noisy GHZ states

$$\rho = v|\text{GHZ}\rangle\langle\text{GHZ}| + (1-v)\frac{1}{2^n}\mathbb{1}. \tag{42}$$

Using the metric, the right-hand side of Proposition 1 for state ρ is given by $(2^n-1)v^2$. In order to find the maximum of the left-hand side over biproduct states, we write

$$L \equiv (T,T^{\text{biprod}})_G = v\sum_{\vec{\mu}\in\mathcal{GHZ}} \text{sgn}\big(T^{\text{GHZ}}_{\vec{\mu}}\big)T^{\text{biprod}}_{\vec{\mu}}, \tag{43}$$

where we introduce a convenient notation for summing over nonzero elements of a "GHZ" metric, and note that after transforming correlation tensors to density operators this reads

$$\begin{aligned} L/v &= 2^{n-1}(\rho_{1,1} + \rho_{1,2^n} + \rho_{2^n,1} + \rho_{2^n,2^n}) - 1 && (44) \\ &= 2^n \text{Tr}\big(\rho^{\text{GHZ}}\rho^{\text{biprod}}\big) - 1. && (45) \end{aligned}$$

In the last step we use $\text{Tr}(\rho^{\text{GHZ}}\rho^{\text{biprod}}) \leqslant \frac{1}{2}$, which holds for all biseparable states [19]. Therefore, $L \leqslant (2^{n-1}-1)v$ and the state ρ is shown to be genuinely multipartite entangled if

$$v > \frac{2^{n-1}-1}{2^n-1}, \tag{46}$$

which is known to be optimal [11,20].

The same metric can reveal genuine multipartite entanglement of many other states. Consider generalized GHZ states

$$|\text{GHZ}_\alpha\rangle = \cos\alpha\,|0\cdots 0\rangle + \sin\alpha\,|1\cdots 1\rangle, \tag{47}$$

with $\alpha \in [0,\frac{\pi}{4}]$. The nonvanishing components of its correlation tensor are given by permutations of indices of the following ones:

$$\begin{aligned} T_{\underbrace{y\cdots y}_{2k}x\cdots x} &= (-1)^k \sin 2\alpha, \quad k = 0,1,\ldots,\lfloor\tfrac{N-1}{2}\rfloor, \\ T_{\underbrace{z\cdots z}_{k}0\cdots 0} &= \begin{cases} 1 & \text{for } k \text{ even} \\ \cos 2\alpha & \text{for } k \text{ odd.} \end{cases} \end{aligned} \tag{48}$$

Taking again the GHZ metric, one can repeat the proof which led to Eq. (46) with the only difference being that now

$$\text{Tr}(\rho^{\text{GHZ}_\alpha}\rho^{\text{biprod}}) \leqslant \cos^2\alpha.$$

In this way we find that the generalized GHZ state mixed with white noise is genuinely multipartite entangled for

$$v > \frac{2^n\cos^2\alpha - 1}{2^n - 1}. \tag{49}$$

Finally, note that this state is fully separable only for $\alpha = 0$. Already for infinitesimally small α it can involve entanglement between all n parties. Clearly, a similar statement would hold for a generalized GHZ state between $n-1$ parties and the nth party having an uncorrelated state. Therefore, in the infinitesimal neighborhood of the state $|0\cdots 0\rangle$ there are states with entanglement between an arbitrary number of subsystems.

VI. CONCLUSIONS

We have presented several sufficient criteria for a multipartite entanglement in the form of nonlinear conditions imposed on correlations of the tested state. The conditions are given in a convenient and simple form, and can be directly applied to given families of entangled states.

An important advantage of our criteria is that in many cases only few definite measurements suffice to detect multiqubit entanglement. Presented criteria are more general than entanglement witnesses due to their nonlinearity. A single new criterion detects a genuine entanglement of many different families of states, whereas one definite witness can detect entanglement of one family of states only.

Here we have given only several examples, however, one can construct infinitely many other ones. Note, that only our conditions with the GHZ metric involved correlations of all qubits as well as only some of them. Note that this is the case for the universal two-qubit entanglement condition given in Ref. [16], thus this seems to be a promising direction for further research. A different series of conditions of such kind, with surprising properties, will be presented elsewhere [21].

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APPENDIX

1. Genuine four-partite entanglement in four-qubit states

a. Exclusion of biseparability

To exclude a biseparability of a four-qubit state, we have to verify the condition (9) for the case of maximizing over $(3+1)$- and $(2+2)$-partition product states:

$$\left(\max_{T^{3+1}}(T^{3+1},T)_G < ||T||^2_G\right) \text{ and } \left(\max_{T^{2+2}}(T^{2+2},T)_G < ||T||^2_G\right) \Longrightarrow T \neq T^{\text{bisep}}, \tag{A1}$$

where G denotes a metric operator in a vector space in which four-qubit correlation tensors are embedded. Using a version of a GHZ metric for a four-qubit system, in which only $T_{\sigma(1122)}$-type terms occur,

$$||\hat{T}||^2_{\text{GHZ}} \equiv T^2_{1111} + T^2_{1122} + T^2_{1221} + T^2_{2211} + T^2_{1212} + T^2_{2121} + T^2_{2112} + T^2_{2222}, \tag{A2}$$

we calculate the first term in Eq. (A1). Since $\hat{T}^{3+1}$ is a pure state, we have $\hat{T}^{3+1} = \hat{T}^3 \otimes \hat{T}^1$. Taking $\hat{T}^1 = \vec{m} = [m_1,m_2,m_3]$, with $\sqrt{m_1^2+m_2^2+m_3^2} = 1$, and assuming $T^3 \otimes T^1$ is an $(ABC+D)$-type product, one obtains

$$\begin{aligned}(\hat{T}^3 \otimes \hat{T}^1,\hat{T}) = {}& T_{111}m_1T_{1111} + T_{112}m_2T_{1122} \\ &+ T_{122}m_1T_{1221} + T_{221}m_1T_{2211} \\ &+ T_{121}m_2T_{1212} + T_{212}m_1T_{2121} \\ &+ T_{211}m_2T_{2112} + T_{222}m_2T_{2222}.\end{aligned} \tag{A3}$$

Due to properties (35)–(37) of a generalized Schmidt decomposition [17] applied now to quantum states, any three-qubit pure state can be expressed as

$$\begin{aligned}|\psi\rangle = {}& \cos(\omega_1)|000\rangle + \cos(\omega_2)\sin(\omega_1)|001\rangle \\ &+ e^{i\phi}\cos(\omega_3)\sin(\omega_1)\sin(\omega_2)|010\rangle \\ &+ \cos(\omega_4)\sin(\omega_1)\sin(\omega_2)\sin(\omega_3)|100\rangle \\ &+ \sin(\omega_1)\sin(\omega_2)\sin(\omega_3)\sin(\omega_4)|111\rangle.\end{aligned} \tag{A4}$$

In this parametrization, terms of $\hat{T}^3$ occurring in Eq. (A3) have the following form:

$$\begin{aligned}&T_{112} = T_{121} = T_{211} = T_{222} = 0,\\ &T_{111} = -T_{122} = -T_{221} = -T_{212} \\ &\quad = \sin(2\omega_1)\sin(\omega_2)\sin(\omega_3)\sin(\omega_4).\end{aligned} \tag{A5}$$

Hence Eq. (A3) simplifies to

$$\begin{aligned}(\hat{T}^3 \otimes \hat{T}^1,\hat{T}) = {}& (T_{1111} - T_{1221} - T_{2211} - T_{2121}) \\ &\times m_1 \sin(2\omega_1)\sin(\omega_2)\sin(\omega_3)\sin(\omega_4).\end{aligned} \tag{A6}$$

The maximization of Eq. (A6) over m_1, ω_1, ω_2, ω_3, and ω_4 is trivial. Let us denote local orthogonal transformations $\hat{O}_1 \otimes \hat{O}_2 \otimes \hat{O}_3 \otimes \hat{O}_4$ as $\hat{O}_{\text{tot}}$. We finally obtain

$$\max_{T^{3+1}}(\hat{T}^{3+1},\hat{T}) = \max_{\hat{O}_{\text{tot}}}|T_{1111} - T_{1221} - T_{2211} - T_{2121}|. \tag{A7}$$

In complete analogy we can find inequalities for other types of $(3+1)$-partition product states ($ABD+C$, $ACD+B$, and $A+BCD$), which leads to the following inequalities:

$$\begin{aligned}&\max_{\hat{O}_{\text{tot}}}|T_{1111} - T_{1221} - T_{2211} - T_{2121}| < ||\hat{T}||^2_{\text{GHZ}},\\ &\max_{\hat{O}_{\text{tot}}}|T_{1111} - T_{2211} - T_{1212} - T_{2112}| < ||\hat{T}||^2_{\text{GHZ}},\\ &\max_{\hat{O}_{\text{tot}}}|T_{1111} - T_{1122} - T_{2121} - T_{2112}| < ||\hat{T}||^2_{\text{GHZ}},\\ &\max_{\hat{O}_{\text{tot}}}|T_{1111} - T_{1122} - T_{1221} - T_{1212}| < ||\hat{T}||^2_{\text{GHZ}}.\end{aligned} \tag{A8}$$

In the case of pure states these inequalities allow us to check if a given state is a $(3+1)$-partition product or not.

Proposition 5. If all the inequalities (A8) hold, then the pure state described by correlation tensor $\hat{T}$ is not a $(3+1)$-partition product.

Now we have to calculate the second element of the conjunction in Eq. (A1) involving maximization over $(2+2)$-partition product states. Since

$$\max_{T^{2+2}}(\hat{T}^{2+2},\hat{T}) = \max_{T^2,T'^2}(\hat{T}^2 \otimes \hat{T}'^2,\hat{T}), \tag{A9}$$

we need to explicitly express $\hat{T}^2 \otimes \hat{T}'^2$. This is very simple due to Eqs. (20):

$$\begin{aligned}&(\hat{T}^2)_{11} = -\sin 2\theta, \quad (\hat{T}^2)_{22} = \sin 2\theta, \quad (\hat{T}^2)_{33} = 1,\\ &(\hat{T}'^2)_{11} = -\sin(2\theta'), \quad (\hat{T}'^2)_{22} = \sin(2\theta'), \quad (\hat{T}'^2)_{33} = 1.\end{aligned} \tag{A10}$$

The only nonvanishing terms of the tensor $\hat{T}^2 \otimes \hat{T}'^2$ are

$$\begin{aligned}&(\hat{T}^2 \otimes \hat{T}'^2)_{1111} = \sin 2\theta \sin 2\theta',\\ &(\hat{T}^2 \otimes \hat{T}'^2)_{1122} = -\sin 2\theta \sin 2\theta',\\ &(\hat{T}^2 \otimes \hat{T}'^2)_{1133} = -\sin 2\theta,\\ &(\hat{T}^2 \otimes \hat{T}'^2)_{2211} = -\sin 2\theta \sin 2\theta',\\ &(\hat{T}^2 \otimes \hat{T}'^2)_{2222} = \sin 2\theta \sin 2\theta',\\ &(\hat{T}^2 \otimes \hat{T}'^2)_{2233} = \sin 2\theta, \quad (\hat{T}^2 \otimes \hat{T}'^2)_{3311} = -\sin 2\theta',\\ &(\hat{T}^2 \otimes \hat{T}'^2)_{3322} = \sin 2\theta', \quad (\hat{T}^2 \otimes \hat{T}'^2)_{3333} = 1.\end{aligned} \tag{A11}$$

Substituting these terms one obtains [in the case of maximizing over $(AB+CD)$-product states]

$$\begin{aligned}&\max_{T^2,T'^2}(\hat{T}^2 \otimes \hat{T}'^2,\hat{T})\\ &\quad = \max_{\hat{O}_{\text{tot}},\theta,\theta'}[\sin 2\theta \sin 2\theta' (T_{1111} - T_{1122} - T_{2211} + T_{2222})\\ &\qquad + \sin 2\theta (T_{2233} - T_{1133})\\ &\qquad + \sin 2\theta' (T_{3322} - T_{3311}) + T_{3333}].\end{aligned} \tag{A12}$$

Since only four terms of Eq. (A11) occur in GHZ metric, expression (A12) has, for maximizing over $(AB+CD)$-type product states, the following simplified form:

$$\max_{T^2,T'^2}(\hat{T}^2\otimes\hat{T}'^2,\hat{T})$$
$$=\max_{\hat{O}_{\rm tot},\theta,\theta'}[\sin 2\theta\,\sin(2\theta')(T_{1111}-T_{1122}-T_{2211}+T_{2222})]$$
$$=\max_{\hat{O}_{\rm tot}}|T_{1111}-T_{1122}-T_{2211}+T_{2222}|. \quad \text{(A13)}$$

Taking into account other types of $(2+2)$-partition product states [that is, of type $(AC+BD)$ and $(AD+BC)$] we obtain the following set of inequalities:

$$\max_{\hat{O}_{\rm tot}}|T_{1111}-T_{1122}-T_{2211}+T_{2222}|<||\hat{T}||^2_{\rm GHZ},$$
$$\max_{\hat{O}_{\rm tot}}|T_{1111}-T_{1212}-T_{2121}+T_{2222}|<||\hat{T}||^2_{\rm GHZ}, \quad \text{(A14)}$$
$$\max_{\hat{O}_{\rm tot}}|T_{1111}-T_{1221}-T_{2112}+T_{2222}|<||\hat{T}||^2_{\rm GHZ}.$$

Finally, we obtain the proposition, which is a direct consequence of condition (A1).

Proposition 6. If all the inequalities (A8) and (A14) hold, then the state described by correlation tensor $\hat{T}$ is genuinely four-partite entangled.

b. Exclusion of 3-separability

Since there is only one type of four-partite three-product state, that is, a $(2+1+1)$-partition product, the condition (9) has the following form:

$$\left(\max_{T^{2+1+1}}(\hat{T}^{2+1+1},\hat{T})_G<||\hat{T}||^2_G\right)\Longrightarrow\hat{T}\neq\hat{T}^{3\ \rm sep}. \quad \text{(A15)}$$

We proceed analogously to the case of excluding biseparability of a three-partite state: $\hat{T}^{2+1+1}=\hat{T}^2\otimes\hat{T}^1\otimes\hat{T}'^1$, and we choose the Schmidt basis for $\hat{T}^2$, in which the only nonvanishing terms are

$$T_{11}=\sin 2\theta,\quad T_{22}=-\sin 2\theta,\quad T_{33}=1, \quad \text{(A16)}$$

while $\hat{T}^1=\vec{m}=[m_1,m_2,m_3]$, with $\sqrt{m_1^2+m_2^2+m_3^2}=1$, and $\hat{T}'^1=\vec{n}=[n_1,n_2,n_3]$, with $\sqrt{n_1^2+n_2^2+n_3^2}=1$. Hence the scalar product $(\hat{T}_\sigma^{2+1+1},\hat{T})$, where σ denotes proper permutation of indices referring to subsystems, can be expressed as

$$(\hat{T}_\sigma^{2+1+1},\hat{T})$$
$$=\sum_{i,j=1}^{3}[(T_{\sigma(11ij)}-T_{\sigma(22ij)})\sin 2\theta+T_{\sigma(33ij)}]m_in_i. \quad \text{(A17)}$$

Now we use the Cauchy-Schwartz inequality

$$\sum_{i,j=1}^{3}[(T_{\sigma(11ij)}-T_{\sigma(22ij)})\sin 2\theta+T_{\sigma(33ij)}]m_in_i$$
$$\leqslant\left(\sum_{i,j=1}^{3}[(T_{\sigma(11ij)}-T_{\sigma(22ij)})\sin 2\theta+T_{\sigma(33ij)}]^2\right)^{1/2}$$
$$\times\left(\sum_{i,j=1}^{3}m_i^2n_j^2\right)^{1/2}$$
$$\leqslant\left(\sum_{i,j=1}^{3}[(T_{\sigma(11ij)}-T_{\sigma(22ij)})\sin 2\theta+T_{\sigma(33ij)}]^2\right)^{1/2}$$
$$\times\left[\left(\sum_{i=1}^{3}m_i^4\right)^{1/2}\left(\sum_{j=1}^{3}n_j^4\right)^{1/2}\right]^{1/2}$$
$$\leqslant\left(\sum_{i,j=1}^{3}[(T_{\sigma(11ij)}-T_{\sigma(22ij)})\sin 2\theta+T_{\sigma(33ij)}]^2\right)^{1/2}. \quad \text{(A18)}$$

The last inequality follows from the Cauchy-Schwartz inequality and the fact that

$$\sum_{i=1}^{3}m_i^2\leqslant 1\Longrightarrow\sum_{i=1}^{3}m_i^4\leqslant 1.$$

From now on we can proceed directly as in the case of maximizing over $(2+1)$-partition product states, obtaining the following proposition.

Proposition 7. If the following inequality holds,

$$\max_{\sigma}\max_{\hat{O}_{\rm tot}}\sqrt{\sum_{i,j=1}^{3}(|T_{\sigma(11ij)}-T_{\sigma(22ij)}|+|T_{\sigma(33ij)}|)^2}<||\hat{T}||^2, \quad \text{(A19)}$$

then the state described by correlation tensor $\hat{T}$ is biseparable or genuinely multiqubit entangled.

[1] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Rev. Mod. Phys. **81**, 865 (2009).
[2] J.-W. Pan, Z.-B. Chen, J.-Y. Lu, H. Weinfurter, A. Zeilinger, and M. Żukowski, Accepted by Rev. Mod. Phys. (2011).
[3] D. Collins, N. Gisin, S. Popescu, D. Roberts, and V. Scarani, Phys. Rev. Lett. **88**, 170405 (2002).
[4] M. Seevinck and J. Uffink, Phys. Rev. A **65**, 012107 (2001).
[5] W. Laskowski and M. Żukowski, Phys. Rev. A **72**, 062112 (2005).
[6] C. Schmid, N. Kiesel, W. Laskowski, W. Wieczorek, M. Żukowski, and H. Weinfurter, Phys. Rev. Lett. **100**, 200407 (2008).
[7] M. Seevinck and J. Uffink, Phys. Rev. A **78**, 032101 (2008).
[8] O. Gühne and G. Tóth, Phys. Rep. **474**, 1 (2009).

[9] B. Jungnitsch, T. Moroder, and O. Gühne, Phys. Rev. Lett. **106**, 190502 (2011).
[10] M. Huber, F. Mintert, A. Gabriel, and B. C. Hiesmayr, Phys. Rev. Lett. **104**, 210501 (2010).
[11] O. Gühne and M. Seevinck, New J. Phys. **12**, 053002 (2010).
[12] P. Hyllus, W. Laskowski, R. Krischek, C. Schwemmer, W. Wieczorek, H. Weinfurter, L. Pezzé, and A. Smerzi, e-print arXiv:1006.4366.
[13] G. Tóth, e-print arXiv:1006.4368.
[14] J. I. de Vicente and M. Huber, e-print arXiv:1106.5756.
[15] P. Badziąg, Č. Brukner, W. Laskowski, T. Paterek and M. Żukowski, Phys. Rev. Lett. **100**, 140403 (2008).
[16] S. Yu, J.-W. Pan, Z.-B. Chen, and Y.-D. Zhang, Phys. Rev. Lett. **91**, 217903 (2003).
[17] H. A. Carteret, A. Higuchi, and A. Sudbery, J. Math. Phys. **41**, 7932 (2000).
[18] W. Laskowski, T. Paterek, Č. Brukner, and M. Żukowski, Phys. Rev. A **81**, 042101 (2010).
[19] A. Acin, D. Bruss, M. Lewenstein, and A. Sanpera, Phys. Rev. Lett. **87**, 040401 (2001).
[20] A. Gabriel, B. C. Hiesmayr, and M. Huber, Quant. Inf. Comput. **10**, 829 (2010).
[21] W. Laskowski, M. Markiewicz, T. Paterek, and M. Żukowski (unpublished).