

ON SERIES OF MULTIQUBIT BELL'S INEQUALITIES

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We review series of multiqubit Bell's inequalities which apply to correlation functions and present conditions that quantum states must satisfy to violate such inequalities.

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1. Introduction

Quantum mechanics gives predictions in form of probabilities. Already some of the fathers of the theory were puzzled with question whether there can exist a deterministic structure beyond quantum mechanics which recovers quantum statistics as averages over “hidden variables” (HV). In his famous impossibility proof Bell made precise assumptions about the form of possible underlying HV structure that allows mutually distant systems to be independent of one another.¹ He derived the inequality which must be satisfied by all such (local realistic) structures and presented the example of quantum predictions which violate it. In this way the famous Einstein–Podolsky–Rosen (EPR) paradox was solved.²

Noncommutativity of quantum theory precludes simultaneous deterministic predictions of measurement outcomes of complementary observables. For the EPR this indicated that “the wave function does not provide a complete description of physical reality”. They expected the complete theory to predict all possible measurement outcomes, prior to and independent of the measurement (realism), and not to allow “spooky action at a distance” (locality). Such a completion was disqualified by Bell.

A more general version of two-particle Bell's theorem was given by Clauser, Horne, Shimony, and Holt (CHSH), and still extended by Clauser and Horne (CH).^{3,4} The important feature of the CHSH and CH inequalities, which hold for all local realistic theories, is that they cannot only be compared with ideal quantum predictions, but also with experimental results. Thus a debate that seemed quasi-philosophical, could be tested in the lab. (see Ref. 5 for review of early experiments).

The three or more particle, or as we now say qubit, versions of Bell's theorem were presented by Greenberger, Horne, and Zeilinger (GHZ), surprisingly 25 years after the original Bell's paper.⁶ In contradistinction with the two-particle case, now the contradiction between local realism and quantum mechanics could be shown for perfect correlations. Immediately after that Mermin produced series of inequalities for arbitrary many particles, which cover the GHZ case, and made the GHZ paradox directly testable in the laboratory.⁷ A complementary series of inequalities was introduced by Ardehali.⁸ In the next step Belinskii and Klyshko gave series of two settings inequalities, which contained the tight inequalities of Mermin and Ardehali.⁹ Finally the full set of tight two setting per observer N -party Bell inequalities for dichotomic observables was found independently in Refs. 10–12. All these series of inequalities are a generalization of the CHSH ones.³ Such inequalities involve only N party correlation functions.

The process of finding new series of Bell inequalities (for many qubits, many settings per observer, involving all possible correlations — for arbitrary experimental arrangement) continues until this day, and is the topic of this overview. We shall try to present the current state-of-the-art, concentrating on multiqubit Bell inequalities for correlation functions. We will *not* give a detailed analysis of the assumptions behind Bell's inequalities. The reader can find them in excellent papers like Refs. 5 and 13.

With the emergence of the new sub-branch of physics (and information theory), Quantum Information, the Bell theorem, and Bell inequalities found applications far away from foundations of quantum physics. The security analysis of the first entanglement based quantum cryptography scheme involves Bell's inequalities.¹⁴ This is now strengthened by the analysis of Scarani and Gisin, who showed that violation of Bell inequality is indeed a valid security criterion in quantum cryptography distribution.¹⁵ Recently it was shown that with every Bell inequality one can associate a specific quantum communication complexity problem. With the use of quantum states which violate such Bell inequality one can always construct a quantum communication complexity problem which outperforms all possible classical ones.¹⁶ This result, as well as the one of Scarani and Gisin, as well as many others, suggest that violation of Bell inequality is a criterion of *direct* usefulness of entanglement in quantum information processing.¹⁷ The authors of this overview think that there are very many open questions associated with future generalizations of series of multiqubit Bell inequalities, and that it is still a fascinating field of studies, which will find new applications, and new surprising results.

2. Bell and Clauser–Horne–Shimony–Holt

The whole history of the EPR paradox can be found in the beautiful review by Clauser and Shimony.⁵ 29 years after the EPR paper Bell proved that the completion of quantum mechanics expected by EPR, is impossible.¹ In his original proof Bell utilized the perfect anticorrelations which arise whenever Alice and Bob measure local spins (with respect to the same direction) on the two-qubit system in the state:

$$|\psi^-\rangle = \frac{1}{\sqrt{2}}[|0\rangle_A|1\rangle_B - |1\rangle_A|0\rangle_B], \quad (1)$$

where $|0\rangle$ and $|1\rangle$ denotes the eigenbasis of the local σ_z operator. However unavoidable experimental imperfections imply that correlations are never perfect. Here we re-derive the CHSH inequality, for which perfect correlations do not have to be assumed and thus the inequality can be directly experimentally checked. The violation of CHSH inequality implies that no local realistic explanation for the observed correlations is possible. But what if the inequality is satisfied? Can we then build a local realistic model for our observations? The answer is negative. Necessary and sufficient condition for local realistic model involves a set of inequalities, not a single one.

The pair emission begins an experimental run. Alice and Bob measure one of the two alternative settings each run. Their choices of what to measure are absolutely free, uncorrelated (statistically independent) with the workings of the source. According to realism all possible measurement outcomes exist prior to and independent of the measurement. Moreover locality assumes that the outcomes of Alice depend on her setting only, and the same for Bob. We denote their predetermined local realistic results as A_1, A_2 for Alice, and B_1, B_2 for Bob. For example if Alice chooses to measure setting “1” she obtains the outcome A_1 , if she chooses to measure “2” she obtains A_2 . Under realism assumption all possible measurement outcomes are defined, even if only some of them are actually measured. The experiments on qubits can give one of two results, to which we ascribe numbers, $+1$ and -1 , i.e. $A_k, B_l = \pm 1$. We form a “vector” out of the predetermined results of each observer: $\mathbf{A} = (A_1, A_2)$ and $\mathbf{B} = (B_1, B_2)$ in this case. One can also define a “vector” (or if you like, a “tensor”) of local realistic correlation functions, \hat{E}_{LR} , with components $E^{LR}(k, l) = \langle A_k B_l \rangle_{\text{avg}}$, where the average is taken over many experimental runs. Then all such local realistic models, \hat{E}_{LR} , can be written as:

$$\hat{E}_{LR} = \sum_{\mathbf{A}, \mathbf{B} = (\pm 1, \pm 1)} P(\mathbf{A}, \mathbf{B}) \mathbf{A} \otimes \mathbf{B}, \quad (2)$$

where $P(\mathbf{A}, \mathbf{B})$ is the probability with which a certain quadruple of predetermined results $\{A_1, A_2, B_1, B_2\}$ appears. That is, every local realistic model of the correlation functions is a convex combination of the extreme points $\mathbf{A} \otimes \mathbf{B}$, and thus lies within a convex polytope, spanned by the vertices $\mathbf{A} \otimes \mathbf{B}$. The necessary and

sufficient condition for local realistic description is a set of inequalities which define the interior of the polytope and are saturated at the border hyperplane of it. Such inequalities are called *tight* Bell's inequalities.

Let us present a construction of the necessary and sufficient condition for the possibility of local realistic description of correlation functions for such experiments with two qubits. First we derive a necessary condition for local realistic model, then construct such a model proving that the condition is also sufficient. We introduce a more elaborated notation for future use in the generalization to arbitrary number of particles. The two local dichotomic observables are parametrized by vectors \mathbf{n}_1^j and \mathbf{n}_2^j , and the predetermined results for the j th party by $A_j(\mathbf{n}_1^j) = \pm 1$ and $A_j(\mathbf{n}_2^j) = \pm 1$, as for now $j = 1, 2$ (1 for Alice, 2 for Bob). Since $A_j(\mathbf{n}_k^j) = \pm 1$, for each observer j one has either $|A_j(\mathbf{n}_1^j) + A_j(\mathbf{n}_2^j)| = 0$ and $|A_j(\mathbf{n}_1^j) - A_j(\mathbf{n}_2^j)| = 2$, or vice versa. Therefore, for all sign choices of $s_1, s_2 = \pm 1$ the product $[A_1(\mathbf{n}_1^1) + s_1 A_1(\mathbf{n}_2^1)][A_2(\mathbf{n}_1^2) + s_2 A_2(\mathbf{n}_2^2)]$ vanishes except for one sign choice, for which it is equal to ± 4 . If one adds up all such four products, with an arbitrary sign in front of each of them, the sum is always equal to the value of the only nonvanishing term, i.e. it is ± 4 . Thus the following algebraic identity holds for the predetermined results:

$$A_{12,12;S} \equiv \sum_{s_1, s_2 = \pm 1} S(s_1, s_2) [A_1(\mathbf{n}_1^1) + s_1 A_1(\mathbf{n}_2^1)][A_2(\mathbf{n}_1^2) + s_2 A_2(\mathbf{n}_2^2)] = \pm 4, \quad (3)$$

where $S(s_1, s_2)$ stands for an arbitrary "sign" function of the summation indices s_1, s_2 , such that its values are only ± 1 . The notation $A_{12,12;S}$ describes the situation where two parties choose between two settings "1" or "2".

After averaging the expression (3) over the ensemble of the runs of the experiment one obtains the following set of Bell inequalities:

$$\left| \sum_{s_1, s_2 = -1, 1} S(s_1, s_2) \sum_{k_1, k_2 = 1, 2} s_1^{k_1-1} s_2^{k_2-1} E(k_1, k_2) \right| \leq 4, \quad (4)$$

where $E(k_1, k_2) = \langle A_1(\mathbf{m}_{k_1}^1) A_2(\mathbf{m}_{k_2}^2) \rangle_{\text{avg}}$ is the correlation function. Since there are 16 different functions $S(s_1, s_2)$, the inequalities (4) represent a set of 16 Bell inequalities for the correlation functions. A specific choice of the sign function, $S(s_1, s_2) = \sqrt{2} \sin(\frac{3}{4}\pi + (s_1 + s_2 - 2)\frac{\pi}{4})$, leads to the well-known CHSH inequality³:

$$|E(1, 1) + E(1, 2) + E(2, 1) - E(2, 2)| \leq 2. \quad (5)$$

The set of all 16 inequalities (4) is equivalent to a *single* general Bell inequality:

$$\sum_{s_1, s_2 = -1, 1} \left| \sum_{k_1, k_2 = 1, 2} s_1^{k_1-1} s_2^{k_2-1} E(k_1, k_2) \right| \leq 4. \quad (6)$$

The equivalence of (6) and (4) is evident, once one recalls that, for real numbers $|a+b| \leq c$ and $|a-b| \leq c$ if and only if $|a|+|b| \leq c$, and writes down a generalization of this property to sequences of an arbitrary length.

Whenever local realistic model exists inequality (6) is satisfied by its predictions. To prove the sufficiency of condition (6) we construct a local realistic model for any set of correlation functions which satisfy it, i.e. we are interested in the local realistic models $E^{LR}(k_1, k_2)$ such that they fully agree the measured correlations $E(k_1, k_2)$ for all possible observables $k_1, k_2 = 1, 2$. Recall that the set of local realistic correlation functions can be put as (2), with $\mathbf{A} = (A_1(\mathbf{n}_1^1), s_1 A_1(\mathbf{n}_2^1))$ and $\mathbf{B} = (A_2(\mathbf{n}_1^2), s_2 A_2(\mathbf{n}_2^2))$. Let us ascribe for fixed s_1, s_2 , a hidden probability that $A_j(\mathbf{n}_1^j) = s_j A_j(\mathbf{n}_2^j)$ (with $j = 1, 2$) in the form similar to Eq. (6):

$$P(s_1, s_2) = \frac{1}{4} \left| \sum_{k_1, k_2} s_1^{k_1-1} s_2^{k_2-1} E(k_1, k_2) \right|. \tag{7}$$

Obviously these probabilities are positive. However they sum up to identity only if inequality (6) is saturated, otherwise there is a “probability deficit”, ΔP . This deficit can be compensated without affecting correlation functions.

First we construct the following structure, which is indeed the local realistic model of the set of correlation functions if the inequality is saturated:

$$\sum_{s_1, s_2 = \pm 1} \Sigma(s_1, s_2) P(s_1, s_2) (1, s_1) \otimes (1, s_2), \tag{8}$$

where $\Sigma(s_1, s_2)$ is the sign of the expression within the modulus in Eq. (7). Now if $\Delta P > 0$ we add a “tail” to this expression given by:

$$\frac{\Delta P}{16} \sum_{\mathbf{A}, \mathbf{B} = (\pm 1, \pm 1)} \mathbf{A} \otimes \mathbf{B}. \tag{9}$$

This “tail” does not contribute to the values of the correlation functions, because it represents the fully random noise. Thus, the sum of (8) is a valid local realistic model for $\hat{E} = (E(1, 1), E(1, 2), E(2, 1), E(2, 2))$.

Let us make some additional remarks. In the four-dimensional real space where both \hat{E}_{LR} and \hat{E} are defined one can find an orthonormal basis set $\hat{S}_{s_1 s_2} = \frac{1}{2}(1, s_1) \otimes (1, s_2)$. Within these definitions the hidden probabilities acquire a simple form:

$$P(s_1, s_2) = \frac{1}{2} |\hat{S}_{s_1 s_2} \cdot \hat{E}|, \tag{10}$$

where the dot denotes the scalar product in \mathcal{R}^4 . Now the local realistic models, \hat{E}_{LR} , can be expressed as:

$$\hat{E}_{LR} = \sum_{s_1, s_2 = \pm 1} |\hat{S}_{s_1 s_2} \cdot \hat{E}| A_1(\mathbf{n}_1^1) A_2(\mathbf{n}_1^2) \hat{S}_{s_1 s_2}. \tag{11}$$

The modulus of any number $|x|$ can be split into $|x| = x \text{sign}(x)$, and we can always demand the product $A_1(\mathbf{n}_1^1) A_2(\mathbf{n}_1^2)$ to have the same sign as the expression inside the modulus. Thus we have:

$$\hat{E} = \sum_{s_1, s_2 = \pm 1} (\hat{S}_{s_1 s_2} \cdot \mathbf{E}) \hat{S}_{s_1 s_2}. \tag{12}$$

The expression in the bracket is the coefficient of tensor \hat{E} in the basis $\hat{S}_{s_1 s_2}$. These coefficients are then summed over the same basis vectors, therefore the last equality appears.

In this way the set of inequalities (4), or its equivalent — the single inequality (6) — is proven to be sufficient and necessary for the possibility of local realistic description of correlation experiments on two qubits, where both Alice and Bob measure one of two local settings. This kind of reasoning can also be applied to arbitrary number of qubits.

3. Many Qubits

Existing features of the GHZ states led to a rapid development of new Bell inequalities for multiqubit systems. Series of Bell inequalities for correlation functions were discovered by Mermin, Ardehali, Belinskii and Klyshko.^{7–9} Below we present a derivation of series which form the complete set of inequalities for N parties, two-settings problem.

The generalization of the approach presented for two-qubit case to many qubits is straightforward. For N particles the generalization of identity (3) consists of the sum of the products $A_j(\mathbf{n}_1^j) + s_j A_j(\mathbf{n}_2^j) = \pm 2$ for the j th party, and the summation is taken with more general “sign function”, $S(s_1, \dots, s_N)$, of N parameters:

$$A_{12, \dots, 12; S} \equiv \sum_{s_1, \dots, s_N = \pm 1} S(s_1, \dots, s_N) \prod_{j=1}^N [A_j(\mathbf{n}_1^j) + s_j A_j(\mathbf{n}_2^j)] = \pm 2^N. \quad (13)$$

Since there are 2^{2^N} different sign functions S , the above formula leads to the set of 2^{2^N} Bell inequalities. Using the trick described above we can write a *single* inequality equivalent to set of all 2^{2^N} inequalities:^{10–12}

$$\sum_{s_1, \dots, s_N = -1, 1} \left| \sum_{k_1, \dots, k_N = 1, 2} s_1^{k_1-1} \dots s_N^{k_N-1} E(k_1, \dots, k_N) \right| \leq 2^N. \quad (14)$$

Many of these inequalities are trivial. e.g., if $S(s_1, \dots, s_N) = 1$ for all arguments, we get the condition $|E(1, 1, \dots, 1)| \leq 1$. In particular, other choices give nontrivial inequalities. For example, for $N = 3$, we can recover the inequality (equivalent to the GHZ argument) given by Mermin⁷:

$$|E(2, 1, 1) + E(1, 2, 1) + E(1, 1, 2) - E(2, 2, 2)| \leq 2. \quad (15)$$

Up to now we have shown that if a local realistic model exists, the general Bell inequality (14) follows. The converse is also true: whenever inequality (14) holds one can construct a local realistic model for the correlation function, in the case of a standard Bell experiment. For N particles the hidden probability that the predetermined outcomes of the j th observer are $A_j(\mathbf{n}_1^j) = s_j A_j(\mathbf{n}_2^j)$ is given by

the form familiar from Eq. (14):

$$P(s_1, \dots, s_N) = \frac{1}{2^N} \left| \sum_{k_1, \dots, k_N=1,2} s_1^{k_1-1} \dots s_N^{k_N-1} E(k_1, \dots, k_N) \right|. \quad (16)$$

The same steps as for two qubits above (now in the \mathcal{R}^{2^N} space) lead to the result that any correlation experiment satisfying (14) can be explained within local realistic picture. This establishes the general Bell inequality (14) as a necessary and sufficient condition for local realistic description of N particle correlation functions in standard Bell-type experiments. That is one can claim that the set of Bell inequalities represented by (14) is complete. This completeness implies that all series of Mermin N -qubit inequalities, which give tight inequalities, are a subset of the inequalities generated by (14). This also applies to tight Ardehali inequalities and the full set of Belinskii-Klyshko inequalities.^{8,9}

4. More Than Two Settings

A general way to establish a necessary and sufficient condition for local realistic description is to define the facets of a correlation polytope.^{19–21} However, this is a computationally hard NP problem.²² Even the complexity of printing out all Bell inequalities grows rapidly with the number of particles or settings involved! Now we present an efficient method for generation of tight Bell's inequalities, which however do not form a complete set. This method was invented by Wu and Zong,^{23,24} and generalized in Ref. 25.

We start with the case of $N = 3$ observers. Suppose that the first two observers can choose between four settings, and the third one between two settings. We denote such problem as $4 \times 4 \times 2$. We already know that the local realistic values satisfy the following algebraic identity:

$$A_{12,12,S'} \equiv \sum_{s_1, s_2=\pm 1} S'(s_1, s_2)[A_1(\mathbf{n}_1^1) + s_1 A_1(\mathbf{n}_2^1)][A_2(\mathbf{n}_1^2) + s_2 A_2(\mathbf{n}_2^2)] = \pm 4, \quad (17)$$

where $S'(s_1, s_2)$ is any sign function, i.e. such that $S'(s_1, s_2) = \pm 1$. In analogous way, we can define $A_{34,34,S''}$ by replacing $A_1(\mathbf{n}_1^1), A_1(\mathbf{n}_2^1), A_2(\mathbf{n}_1^2), A_2(\mathbf{n}_2^2)$ by $A_1(\mathbf{n}_3^1), A_1(\mathbf{n}_4^1), A_2(\mathbf{n}_3^2), A_2(\mathbf{n}_4^2)$, respectively, and S' by S'' . Depending on the value of $s = \pm 1$ one has $(A_{12,S'} + s A_{34,S''}) = \pm 8$, or 0. By analogy to (17) one has:

$$A_{1234,12} \equiv \sum_{s_1, s_2=\pm 1} S(s_1, s_2)[A_{12,S'} + s_1 A_{34,S''}][A_3(\mathbf{n}_1^3) + s_2 A_3(\mathbf{n}_2^3)] = \pm 16. \quad (18)$$

After averaging over many runs of the experiment, and introducing the correlation functions $E(i, j, k) \equiv \langle A_1(\mathbf{n}_i^1) A_2(\mathbf{n}_j^2) A_3(\mathbf{n}_k^3) \rangle_{\text{avg}}$ one obtains multisetting Bell's inequalities. Because of the freedom to choose the sign functions S, S', S'' , we have $(2^4)^3 = 2^{12}$ Bell's inequalities.

All these inequalities can be reduced to single “generating” inequality, in which all the sign functions S, S', S'' are non-factorable. The choice of factorable sign function is equivalent to having a non-factorable one, and some of the local measurement settings equal.²⁵ The “generating” Bell’s inequality can be chosen as (here all sign functions are equal to $\sqrt{2} \sin(\frac{3}{4}\pi + (s_1 + s_2 - 2)\frac{\pi}{4})$):

$$\begin{aligned} & | \langle [A_3(\mathbf{n}_1^3) + A_3(\mathbf{n}_2^3)][A_1(\mathbf{n}_1^1)[(A_2(\mathbf{n}_1^2) + A_2(\mathbf{n}_2^2))] + A_1(\mathbf{n}_2^1)[A_2(\mathbf{n}_1^2) - A_2(\mathbf{n}_2^2)] \\ & + [A_3(\mathbf{n}_1^3) - A_3(\mathbf{n}_2^3)][A_1(\mathbf{n}_3^1)[A_2(\mathbf{n}_3^2) + A_2(\mathbf{n}_4^2)] + A_1(\mathbf{n}_4^1)[A_2(\mathbf{n}_3^2) - A_2(\mathbf{n}_4^2)] \rangle_{\text{avg}} | \\ & \leq 4. \end{aligned} \quad (19)$$

All other inequalities can be obtained by changing $A_j(\mathbf{n}_k^j) \rightarrow -A_j(\mathbf{n}_k^j)$.

The method can be generalized to various choices of the number of parties and the measurement settings. Now we present the $2^{N-1} \times 2^{N-1} \times 2^{N-2} \times \dots \times 2$ case. Consider $N = 4$ observers. We start with the identity (18). One can introduce a similar formula for the settings $\{5, 6, 7, 8\}$, for the first two observers, and $\{3, 4\}$, for the third one. The fourth observer chooses between two settings with local realistic values $A_4(\mathbf{n}_1^4)$ and $A_4(\mathbf{n}_2^4)$. Applying the same method as before, one obtains an identity which generates Bell’s inequalities of the $8 \times 8 \times 4 \times 2$ type:

$$\sum_{s_1, s_2 = \pm 1} S(s_1, s_2)[A_{1234,12} + s_1 A_{5678,34}][A_4(\mathbf{n}_1^4) + s_2 A_4(\mathbf{n}_2^4)] = \pm 64, \quad (20)$$

where $A_{1234,12}$ and $A_{5678,34}$ depend on some three sign functions. One may apply this method iteratively, increasing the number of observers by one, to obtain inequalities involving exponential (in N) number of measurement settings.

As another example we construct the inequalities involving N partners, where first $N - 1$ observers choose one of four settings and the last one chooses between two settings. We use the local realistic quantity $A_{12, \dots, 12}$ defined in Eq. (13) for $N - 1$ parties choosing between two settings each:

$$A_{12, \dots, 12} \equiv \sum_{s_1, \dots, s_{N-1} = \pm 1} S'(s_1, \dots, s_{N-1}) \prod_{j=1}^{N-1} [A_j(\mathbf{n}_1^j) + s_j A_j(\mathbf{n}_2^j)] = \pm 2^{N-1}, \quad (21)$$

and analogically introduce $A_{34, \dots, 34}$ for another pair of observables available to each party. The sign function in $A_{34, \dots, 34}$ can be different from $S'(s_1, \dots, s_{N-1})$ in Eq. (21). By including the N th observer, who can choose between two measurement settings, we obtain:

$$\sum_{s_1, s_1 = \pm 1} S(s_1, s_2)(A_{12, \dots, 12} + s_1 A_{34, \dots, 34})(A_N(\mathbf{n}_1^N) + s_2 A_N(\mathbf{n}_2^N)) = \pm 2^{N+1}. \quad (22)$$

One can use this expression for generating Bell inequalities for N observers in the same way as it was previously done.

In order to show the full strength of the method our next example gives a family of Bell inequalities for $N = 5$ qubits, which involves eight settings for first two observers and four settings for the other three. We take the identity $A_{1234,12}$

defined in (18), valid for the $4 \times 4 \times 2$ case of three observers, and define a similar quantity for another set of $4 \times 4 \times 2$ observables, namely $A_{5678,34}$. Note that the sign functions entering $A_{5678,34}$ can be different from those entering $A_{1234,12}$. For the other two observers we introduce:

$$A_{12,12} \equiv \sum_{s_1, s_2 = \pm 1} S'(s_1, s_2)(D_1 + s_1 D_2)(E_1 + s_2 E_2) = \pm 4 \quad (23)$$

and a similar expression for another pair of observables D_3, D_4 and E_3, E_4 :

$$A_{34,34} = \sum_{s_1, s_2 = \pm 1} S''(s_1, s_2)(D_3 + s_1 D_4)(E_3 + s_2 E_4) = \pm 4. \quad (24)$$

In the next step we get the following algebraic identity which can be used, via averaging, to generate a family of Bell inequalities:

$$\sum_{s_1, s_2 = \pm 1} S(s_1, s_2)(A_{1234,12} + s_1 A_{5678,34})(A_{12,12} + s_2 A_{34,34}) = \pm 256. \quad (25)$$

It is clear that there is no bound in extending this type of derivations. Finally let us note that all the inequalities with lower number of settings can be obtained from our construction by making some of the local settings identical.

The multisetting inequalities constructed by the above procedure are tight. Consider the case of $4 \times 4 \times 2$ inequalities. The left-hand side of the identity (18) is equal to ± 16 for any combination of predetermined local realistic results. In a 32-dimensional real space, one can build a convex polytope, containing all possible local realistic models of the correlation functions for the specified settings, with vertices given by the tensor products of $\hat{v} = (A_1(\mathbf{n}_1^1), A_1(\mathbf{n}_2^1), A_1(\mathbf{n}_3^1), A_1(\mathbf{n}_4^1)) \otimes (A_2(\mathbf{n}_1^2), A_2(\mathbf{n}_2^2), A_2(\mathbf{n}_3^2), A_2(\mathbf{n}_4^2)) \otimes (A_3(\mathbf{n}_1^3), A_3(\mathbf{n}_2^3))$. It has 256 different vertices. Tight Bell inequalities define the half-spaces in which is the polytope, which contain a face of it in their border hyperplane. If 32 linearly independent vertices belong to a hyperplane, this hyperplane defines a tight inequality. Half of the vertices give in (18) the value 16 and another half give -16 . Every vertex, \hat{v} from the first set has a partner $-\hat{v}$ in the second one. Next notice that any set of 128 vertices \hat{v} , which does not contain pairs \hat{v} and $-\hat{v}$ contains a set of 32 linearly independent points (basis). Thus, each inequality is tight. This reasoning can be adapted to all inequalities discussed here.

Most importantly, the multisetting inequalities reveal violation of local realism of classes of states, for which standard inequalities, with two measurement settings per side, are satisfied.

5. Quantum Violations

Bell inequalities are interesting only if there exists a certain state which violates them (for certain measurement settings). Let us derive the condition for violation of the general Bell inequality involving two measurement settings, (14), by an arbitrary two-qubit state. This result is a generalization and reformulation of the condition

given for two qubits by the Horodecki Family.²⁶ We will use a decomposition of general mixed state of N qubits in terms of the identity operator $\sigma_0 = I$ and the Pauli operators σ_i for three orthogonal directions $i \in \{1, 2, 3\}$, given by:

$$\rho = \frac{1}{2^N} \sum_{k_1, \dots, k_N=0}^3 T_{k_1 \dots k_N} \sigma_{k_1} \otimes \dots \otimes \sigma_{k_N}, \quad (26)$$

where all the σ_i operators act in the Hilbert space of individual qubits. The (real) coefficients $T_{k_1 \dots k_N}$, with $k_j = 1, 2, 3$, form the correlation tensor \hat{T} . Note further that $T_{k_1 \dots k_N} = \text{Tr}(\rho \sigma_{k_1} \otimes \dots \otimes \sigma_{k_N})$, i.e. all coefficients are directly experimentally accessible.

The full set of inequalities for the 2×2 problem is derivable from the identity (17) where we put non-factorable sign function:

$$|\langle (A_1 + A_2)B_1 + (A_1 - A_2)B_2 \rangle_{\text{avg}}| \leq 2. \quad (27)$$

All other inequalities can be obtained by all possible sign changes $X_k \rightarrow -X_k$ (with $k = 1, 2$ and $X = A, B$). The quantum correlation function $E(\mathbf{a}_k, \mathbf{b}_l)$ is given by the scalar product of the correlation tensor \hat{T} with the tensor product of the local measurement settings represented by unit vectors $\mathbf{a}_k \otimes \mathbf{b}_l$, i.e. $E(\mathbf{a}_k, \mathbf{b}_l) = (\mathbf{a}_k \otimes \mathbf{b}_l) \cdot \hat{T}$. Thus, the condition for a quantum state endowed with the correlation tensor \hat{T} to satisfy the inequality (27), is that for all directions $\mathbf{a}_1, \mathbf{a}_2, \mathbf{b}_1, \mathbf{b}_2$ one has

$$\left| \left[\left(\frac{\mathbf{a}_1 + \mathbf{a}_2}{2} \right) \otimes \mathbf{b}_1 + \left(\frac{\mathbf{a}_1 - \mathbf{a}_2}{2} \right) \otimes \mathbf{b}_2 \right] \cdot \hat{T} \right| \leq 1, \quad (28)$$

where both sides of (27) were divided by 2.

Note that the pairs of local vectors define the ‘‘local measurement planes’’. Here we shall find the conditions for (28) to hold for two, arbitrary but fixed, measurement planes, one for each observer. Therefore only those components of \hat{T} are relevant which describe measurements in these two planes. Thus \hat{T} is effectively described by a 2×2 matrix, or tensor \hat{T}' .

Notice that $\mathbf{A}_\pm = \frac{1}{2}(\mathbf{a}_1 \pm \mathbf{a}_2)$ satisfy the following relations: $\mathbf{A}_+ \cdot \mathbf{A}_- = 0$ and $\|\mathbf{A}_+\|^2 + \|\mathbf{A}_-\|^2 = 1$. Thus $\mathbf{A}_+ + \mathbf{A}_-$ is a unit vector, and \mathbf{A}_\pm represent its decomposition into two orthogonal vectors. Thus if one introduces unit vectors \mathbf{a}_\pm such that $\mathbf{A}_\pm = a_\pm \mathbf{a}_\pm$, one has $a_+^2 + a_-^2 = 1$. Thus one can put (28) into the following form:

$$|\hat{S} \cdot \hat{T}'| \leq 1, \quad (29)$$

where $\hat{S} = a_+ \mathbf{a}_+ \otimes \mathbf{b}_1 + a_- \mathbf{a}_- \otimes \mathbf{b}_2$. Since $\mathbf{a}_+ \cdot \mathbf{a}_- = 0$, one has $\hat{S} \cdot \hat{S} = 1$, i.e. \hat{S} is a tensor of unit norm. Any tensor of unit norm, \hat{U} , has the following Schmidt decomposition $\hat{U} = \lambda_1 \mathbf{v}_1 \otimes \mathbf{w}_1 + \lambda_2 \mathbf{v}_2 \otimes \mathbf{w}_2$, where $\mathbf{v}_i \cdot \mathbf{v}_j = \delta_{ij}$, $\mathbf{w}_i \cdot \mathbf{w}_j = \delta_{ij}$ and $\lambda_1^2 + \lambda_2^2 = 1$. The (complete) freedom of the choice of the measurement directions \mathbf{b}_1 and \mathbf{b}_2 , allows one, by choosing \mathbf{b}_2 orthogonal to \mathbf{b}_1 , to put \hat{S} in the form isomorphic with \hat{U} . The freedom of choice of \mathbf{a}_1 and \mathbf{a}_2 allows \mathbf{A}_+ and \mathbf{A}_- to be

arbitrary orthogonal unit vectors, and \mathbf{a}_+ and \mathbf{a}_- to be also arbitrary. Thus \hat{S} can be equal to any unit tensor. Therefore to get the maximum of the left-hand side of (28) we put $\hat{S} = \frac{1}{\|\hat{T}'\|} \hat{T}'$, and the maximum is $\|\hat{T}'\| = \sqrt{\hat{T}' \cdot \hat{T}'}$. Thus,

$$\max \left[\sum_{k,l=1,2} T_{kl}^2 \right] \leq 1, \tag{30}$$

where the maximization is taken over all local coordinate systems of two observers, is the necessary and sufficient condition for the inequality (14) to hold for quantum predictions.

For $2^{N-1} \times 2^{N-1} \times \dots \times 2$ multisetting inequalities we can derive similar simple conditions for larger number of qubits. Consider the case of three qubits. For this situation we have the following inequality:

$$\left| \left\langle \sum_{s_1, s_2} S(s_1, s_2) [A_{12, S'} + s_1 A_{34, S''}] [C_1 + s_2 C_2] \right\rangle_{\text{avg}} \right| \leq 16, \tag{31}$$

where S, S', S'' are some non-factorable sign functions. The three-qubit quantum correlation functions $E(\mathbf{a}_i, \mathbf{b}_j, \mathbf{c}_k)$ can be represented as $(\mathbf{a}_i \otimes \mathbf{c}_j \otimes \mathbf{c}_k) \cdot \hat{T}$ (with the same meaning of the symbols as before; \hat{T} is now a three-index tensor). Thus the condition for the $4 \times 4 \times 2$ inequalities to hold, in the quantum case, transforms into

$$|[\hat{A}_{12, S'} \otimes (\mathbf{c}_1 + \mathbf{c}_2) + \hat{A}_{34, S''} \otimes (\mathbf{c}_1 - \mathbf{c}_2)] \cdot \hat{T}| \leq 8, \tag{32}$$

where e.g.

$$\hat{A}_{12, S'} = \sum_{s_1, s_2 = \pm 1} S'(s_1, s_2) (\mathbf{a}_1 + s_1 \mathbf{a}_2) \otimes (\mathbf{b}_1 + s_2 \mathbf{b}_2). \tag{33}$$

To write down (32) we have used the freedom of introducing the sign changes $\mathbf{X}_i \rightarrow -\mathbf{X}_i$, compare (19). By defining $\frac{1}{2}(\mathbf{c}_1 \pm \mathbf{c}_2) = \mathbf{c}_\pm$ which have the similar properties as a_\pm and \mathbf{a}_\pm , inequality (32) transforms to:

$$|c_+ \hat{A}_{12, S'} \otimes \mathbf{c}_+ \cdot \hat{T} + c_- \hat{A}_{34, S''} \otimes \mathbf{c}_- \cdot \hat{T}| \leq 4. \tag{34}$$

One can always choose c_+ and c_- that maximize the left-hand side. Since $c_+^2 + c_-^2 = 1$ this leads to the condition:

$$[\hat{A}_{12, S'} \cdot \hat{T}^{(+)}]^2 + [\hat{A}_{34, S''} \cdot \hat{T}^{(-)}]^2 \leq 4^2, \tag{35}$$

where $\hat{T}^{(\pm)}$ is defined by $T_{ij}^{(\pm)} = \sum_{k=1}^3 (c_\pm)_k T_{ijk}$, where in turn $(c_\pm)_k$ is the k th component of vector \mathbf{c}_\pm . Note that since \mathbf{c}_+ and \mathbf{c}_- are orthogonal and normalized, this procedure amounts to fixing of two (new) Cartesian axes for the third observer, and accordingly transforming the correlation tensor. Since $\hat{A}_{12, S'}$ depends on different vectors than $\hat{A}_{34, S''}$, one can maximize the two terms separately. Furthermore, since the problem of maximization of $\hat{A}_{nm, S} \cdot \hat{T}^{(\pm)}$ is equivalent to the 2×2 case

Table 1. The examples of necessary and sufficient conditions for violation of multisetting inequalities.

N	Case	C_N (the condition)
2	2×2	$\sum_{k,l=1,2} T_{kl}^2 \leq 1$
3	$4 \times 4 \times 2$	$\sum_{k,l=1,2} T_{kl2}^2 + \sum_{k',l'=1,2} T_{k'l'1}^2 \leq 1$
4	$8 \times 8 \times 4 \times 2$	$\begin{aligned} &\sum_{k_1,l_1=1,2} T_{k_1l_122}^2 + \sum_{k_2,l_2=1,2} T_{k_2l_212}^2 \\ &+ \sum_{k_3,l_3=1,2} T_{k_3l_321}^2 + \sum_{k_4,l_4=1,2} T_{k_4l_411}^2 \leq 1 \end{aligned}$

studied earlier, the overall maximization process gives the following *necessary and sufficient* condition for quantum correlations to satisfy the inequality (31):

$$\max \sum_{x=1,2} \sum_{k_x,l_x=1,2} T_{k_xl_x x}^2 \leq 1. \tag{36}$$

When compared with the *sufficient* condition for $2 \times 2 \times 2$ inequalities to hold, namely¹²:

$$\max \left[\sum_{k,l,m=1,2} T_{klm}^2 \right] \leq 1, \tag{37}$$

the new condition is *more demanding* because the Cartesian coordinate systems denoted by the indices k_1, l_1 and k_2, l_2 do not have to be the same. Note further that if one uses in both terms in (36) *the same planes of observations* (for the first two observers) then the condition reduces to (37), which now is a sufficient and necessary one (but for $4 \times 4 \times 2$ settings). Thus the condition is necessary and sufficient whenever the vectors defining measurements for each party are limited to one plane.

In a similar way one can reach analogous conditions for violation of $2^{N-1} \times 2^{N-1} \times 2^{N-2} \times \dots \times 2$ inequalities by quantum predictions. In Table 1 we present these conditions for small N . One can note a useful recurrence that can be used to write down the condition for arbitrary N . Let us define:

$$\mathcal{C}_2 \equiv \sum_{k,l=1,2} T_{kl}^2. \tag{38}$$

Then the condition for two qubits reads: $\max(\mathcal{C}_2) \leq 1$. Next let us put a recursive definition:

$$\mathcal{C}_N = [\mathcal{C}_{N-1}]_{\oplus 2} + [\mathcal{C}_{N-1}]'_{\oplus 1}, \tag{39}$$

where $[\mathcal{C}_{N-1}]_{\oplus k}$ is the expression in the condition for $N - 1$ qubits in which the correlation tensor elements $T_{i_1 \dots i_{N-1}}$ are replaced by $T_{i_1 \dots i_{N-1} k}$, i.e. elements of the

N -qubit correlation tensor. The “prime” denotes the fact that the second term does not involve components of \hat{T} in the same set of coordinate systems (for the first $N - 1$ observers) as the unprimed term.

The sufficient and necessary condition for N qubits to satisfy all $2^{N-1} \times 2^{N-1} \times 2^{N-2} \times \dots \times 2$ inequalities, within this convention reads:

$$\max(\mathcal{C}_N) \leq 1. \quad (40)$$

6. Gisin's Problem

The theorem of Gisin states that *any* pure non-product state violates local realism, i.e. there are sets of measurements that can be performed on the state which cannot be described within local realistic picture.^{27,28} This theorem formalizes the intuition that entanglement is a purely quantum phenomenon. Using the approach presented here we can write down the following proof of Gisin's theorem for *two qubits*. Any state of two qubits is given in its Schmidt basis by $|\psi\rangle = \cos\alpha|00\rangle + \sin\alpha|11\rangle$, with $\alpha \in [0, \pi/4]$. The correlation tensor of this state has the following coefficients: $T_{xx} = \sin 2\alpha$, $T_{yy} = -\sin 2\alpha$, $T_{zz} = 1$. Therefore the necessary and sufficient condition for local realism is violated for all non-product ($\alpha \neq 0$) states:

$$\sum_{k,l=\{x,z\}} T_{kl}^2 = 1 + \sin^2 2\alpha > 1. \quad (41)$$

As we shall show here the series of the two-settings inequalities (14) for N qubits are failing to show violation of local realism for an important class of entangled pure states. This looks, superficially, as a counterexample to Gisin's theorem. But the theorem uses all possible measurement scenarios, which is not the case in the case of inequalities (14). Further, as it will be shown, the more-than-two setting inequalities do show violations for this class of states. This is a strong argument for continuation of the research towards finding inequalities with even more general structure.

Scarani and Gisin noticed a surprising feature of the following state²⁹:

$$|\psi\rangle = \cos\alpha|0 \dots 0\rangle + \sin\alpha|1 \dots 1\rangle. \quad (42)$$

They showed that for $\sin 2\alpha \leq 1/\sqrt{2^{N-1}}$ the states (42) do not violate the Mermin–Ardehali–Belinskii–Klyshko (MABK) inequalities.^{7–9} This has been numerically obtained for $N = 3, 4, 5$ and conjectured for $N > 5$. Their result contrasts the case of two qubits and is highly counterintuitive as the states (42) are generalization of the GHZ states, which violate maximally the MABK inequalities.⁶ The problem of Scarani and Gisin was later analyzed using the general two-qubit correlation function Bell inequality for arbitrary number of qubits (14).³⁰ The family (42) does not violate *any* such Bell inequality if α and N are such that $\sin 2\alpha \leq 1/\sqrt{2^{N-1}}$ and N is *odd*. However, for even number of qubits the general Bell inequality imposes stronger constraints for local realistic description, than MABK inequalities.

What are the reasons for the completely different behavior for N even and N odd? The expression $\sum_{k_1, \dots, k_N=x,y} T_{k_1 \dots k_N}^2$, which appears in the condition for

violation of two-setting inequalities, can be understood as a “total measure of the strength of correlations” in mutually complementary sets of local measurements (as defined by the summation over x and y).³¹ Then the unity on the right-hand side of the condition is the classical limit for the amount of correlations. Specifically, pure product states cannot exceed the limit of 1, as they can show perfect correlations in one set of local measurement directions only. In contrast, entangled states can show perfect correlations for more than one such set.³¹ Now, only if N is even, the states (42) already show perfect correlation between measurements along z -directions (as the product is then always $+1$) reaching therefore the classical limit (we assume σ_z eigenstates give the qubit computational basis). Yet, they also show additional correlations in other, complementary, directions. In the case of N odd, however, there are no perfect correlations along z -direction and the correlations in the complementary directions do not suffice to violate the bound of 1.

Since the condition for multipartite two-setting inequalities to hold:

$$\max_{k_1, \dots, k_N=1,2} \sum T_{k_1 \dots k_N}^2 \leq 1, \quad (43)$$

is only necessary, one cannot draw any conclusions about its possible local realistic model for a state satisfying this condition. Thus it could be that the states (42) can violate local realism if we use some more optimal tool to test it. Indeed multisetting inequalities described extend the class of quantum states which do not admit local realistic explanation. Also the generalized GHZ states cannot be classically explained. Their nonvanishing correlation tensor components are (directions 1, 2, 3 are denoted by x, y, z ; the basis $\{|0\rangle, |1\rangle\}$ is the eigenbasis of σ_z): $T_{z \dots z} = \cos 2\alpha$, for N odd, and 1 for N even, $T_{x \dots x} = \sin 2\alpha$, and the components with $2k$ indices equal to y and the rest equal to x take the value $(-1)^k \sin 2\alpha$ (there are 2^{N-2} such components). Let us assume that the last observer can choose only between settings x and z . Thus, we obtain for the condition for violation of multisetting Bell’s inequality for *odd* number of observers (generalization of the condition (36))

$$\begin{aligned} & \sum_{k_1, \dots, k_{N-1}=x,y} T_{k_1 \dots k_{N-1}x}^2 + \sum_{k_1, \dots, k_{N-1}=x,z} T_{k_1 \dots k_{N-1}z}^2 \\ & = 2^{N-2} \sin^2 2\alpha + \cos^2 2\alpha > 1. \end{aligned} \quad (44)$$

Thus, the Bell’s inequalities are violated for the whole range of $\pi/4 \geq \alpha > 0$ and for arbitrary N in contrast to the case of standard Bell’s inequalities.

7. Prospects

We have described some series of Bell’s inequalities for arbitrary number of qubits involving N -particle correlation functions. In the case of two settings per observer the inequalities form a complete set, which gives the necessary and sufficient condition for local realistic description of correlation experiments. Even for this simple

scenario it is still an open question how would the necessary and sufficient condition look like for more general case involving all correlations of all ranks between observers, such as those described by correlation functions for $N - 1$ parties, etc. (generalizations of CH inequality). The other problem is to consider more settings per observer. A step towards this direction was recently made by one of us, who constructed a full set of inequalities for three settings per observer and arbitrary number of parties.³² The final aim is to construct such inequalities for arbitrary experimental setup.

It is clear that only entangled states can violate Bell's inequalities, as any separable state has a local realistic model. For pure entangled states there always exists Bell's inequality that is violated by certain measurements performed on the state. But which mixed states violate local realism? Perhaps a step forward could be achieved by the introduction of other, additional to Bell's (local realism) assumptions, based on the fundamental symmetries of physical laws, which would narrow the class of physically admissible local realistic theories.^{33,34}

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